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FORMULATION OF FOUR POLES OF THREE-DIMENSIONAL ACOUSTIC CAVITIES USING PRESSURE RESPONSE FUNCTIONS WITH SPECIAL ATTENTION TO SOURCE MODELING

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Abstract

The procedure proposed by Kim and Soedel [1] for formulation of four pole parameters of three-dimensional cavities is revised. In the procedure, four poles were formulated in terms of the pressure responses of the cavities to a point source. However, it is shown that using the point source model for such a purpose is not valid because the pressure response function becomes singular at the source point. In this work, the procedure is modified by employing a surface source. It is shown that the modified procedure can be applied to three-dimensional acoustic systems.

Introduction

A four pole matrix is a very convenient concept to analyze complex acoustic systems. It allows various acoustic elements of the system to be formulated independently and to be assembled to form the system equation. Also, using four poles reduces related analysis efforts substantially as the system equation remains a two-by-two matrix. Many applications are found for analysis of one-dimensional systems [2, 3] and lumped parameter systems [2]. It is very appealing to have four poles of three-dimensional cavities because they are easily integrated with those of one-dimensional acoustic cavities for the purpose of system analysis. Kim and Soedel [1] proposed a method to formulate four pole parameters of three-dimensional cavities in terms of the pressure response functions of the systems at the input and output points. Lai and Soedel [4] applied a similar concept to analyze shallow three-dimensional cavities by specializing the procedure for two-dimensional cases. In all these works [1, 4], pressure response functions were obtained by solving the wave equation of cavities and modeling flow input and output ports as point sources.

Deriving a four pole matrix of a three-dimensional cavity implies that the cavity is connected to one-dimensional systems. Therefore, the size of its mass flow source can generally be considered much smaller than other dimensions of the cavity. Hence, it appears to be logical to model an acoustic source as a point source, as it has been done in [1, 4, 5]. However, in this work, it is shown that the point source model cannot be used to derive four poles of three- or two-dimensional cavities because of the singularity at the source point. The alternative approach is to use the surface source model. It is shown that an extended model can be generated by connecting one-dimensional pipes of proper lengths to three-dimensional cavities, which enables the four poles of such a sub-system to be derived. Numerical examples are shown, where the boundary element method is used for actual calculations, to illustrate how the concept is used for system analysis.

Formulation of Four Poles using Pressure Response Functions

A four pole matrix defines the relationship between the input and output variables of an acoustic system in the frequency domain. For the acoustic system shown in Figure 1, the equation is defined as:

$$\begin{Bmatrix} Q_1 \\ p_1 \end{Bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{Bmatrix} Q_2 \\ p_2 \end{Bmatrix}, \qquad (1)$$

where P and Q are the amplitudes of the acoustic pressure and volume flow rate, subscripts 1 and 2 indicate the input and output points, respectively, A, B, C and D are the four pole parameters. It was shown that four pole parameters of an acoustic system could be formulated from the pressure response functions of the system as follows [1]:

$$A = \frac{f_{22}(\omega)}{f_{12}(\omega)},\quad B = \frac{1}{f_{12}(\omega)},\quad C = -f_{21}(\omega) + \frac{f_{11}(\omega)}{f_{12}(\omega)} f_{22}(\omega),\quad D = \frac{f_{11}(\omega)}{f_{12}(\omega)}, \qquad (2)$$

where ω is the circular frequency and $f_{ij}(\omega)$ is defined as the pressure response of the system at location i when the system is subjected to a single harmonic volume flow input with unit strength at location j.

According to the definition, pressure response functions used in the four pole formulation have units of pressure per volume flow. In the SI system, for example, the unit becomes *Pa.s/m[3]*. From the acoustic reciprocal principle, it can be easily shown that

$$f_{12}(\omega) = f_{21}(\omega)\,. \tag{3}$$

Therefore, four pole parameters of any general acoustic systems are derived if three pressure response functions, $f_{11}(\omega)$, $f_{22}(\omega)$, and $f_{21}(\omega)$ or $f_{12}(\omega)$, are available.

According to equation (2), four poles can be easily formulated, as long as the all pressure response functions, $f_{ij}(\omega)$, have unique and convergent values. In this work, it is shown that two-dimensional surface sources must be used for the calculation of four poles of a three-dimensional cavity, and one-dimensional line sources must be used in a two-dimensional cavity to satisfy the above conditions. Also, it is shown that point sources can be only used for one-dimensional systems. In the following, a rectangular cavity is used as an example to explain these concepts because an exact solution of pressure response is available using the modal superposition method.

Pressure Response in a Rectangular Cavity

For a rectangular cavity shown in Figure 2 or 3, the natural mode P_{lmn} and the natural frequency $k_{lmn} = \omega_{lmn}/c_0$, where c_0 is speed of sound, are [6],

$$\begin{cases} P_{lmn}(x,y,z) = \cos\dfrac{lx}{L_x}\cos\dfrac{my}{L_y}\cos\dfrac{nz}{L_z} \\ k_{lmn} = \pi[(\dfrac{l}{L_x})^2 + (\dfrac{m}{L_y})^2 + (\dfrac{n}{L_z})^2]^{\frac{1}{2}} \end{cases} \quad l,\ m,\ n = 0,\ 1,\ 2,\ \cdots, \tag{4}$$

Using the modal superposition method, pressure response at any point in the cavity due to a distributed mass flow source, $\dot{M}(x,y,z,\omega)$, becomes

$$p(x,y,z)e^{j\omega t} = -\sum_{l,m,n=0}^{\infty} \frac{j\omega P_{lmn}(x,y,z)\int_{\Omega} \dot{M}(\bar{x},\bar{y},\bar{z},\omega)P_{lmn}(\bar{x},\bar{y},\bar{z})d\bar{x}d\bar{y}d\bar{z}}{(k^2-k_{lmn}^2)\int_{\Omega} P_{lmn}^2(\bar{x},\bar{y},\bar{z})d\bar{x}d\bar{y}d\bar{z}} e^{j\omega t}, \tag{5}$$

where $k = \omega/c_0$, Ω is the whole acoustic domain, and the overbar denotes integration variables.

Mass flow source due to a point source located at $\mathbf{r}_s = (x_s, y_s, z_s)$ is described as

$$\dot{M}(x,y,z,\omega) = \rho_0 Q\delta(x-x_s)\delta(y-y_s)\delta(z-z_s)\,, \tag{6}$$

where ρ_0 is the mean density of acoustic medium, $\mathbf{r}_s$ is the coordinates of the source point, Q_0 is the volume flow harmonic amplitude, and $\delta(\cdot)$ is the Dirac Delta function.

As shown in Figures 3 and 4, a rectangular surface source is considered. This model may be used to approximate a port in a three-dimensional cavity connected to a small rectangular pipe. Further, by assuming that the mass flow distribution on the source surface is constant, the mass flow source is expressed as

$$\begin{aligned} \dot{M}(x,y,z,\omega) = \rho_0 \frac{Q_0}{b_s c_s}\delta(x-x_s)[H(y-y_s+\frac{b_s}{2}) - H(y-y_s-\frac{b_s}{2})] \\ \times[H(y-z_s+\frac{c_s}{2}) - H(y-z_s-\frac{c_s}{2})]. \end{aligned} \tag{7}$$

where $\mathbf{r}_s = (x_s, y_s, z_s)$ is the coordinates of the center of the source surface, b_s and c_s indicate the size of the rectangular source surface, and $H(\cdot)$ is the unit step function.

By letting $(x_1, y_1, z_1) = (x_s, y_s, z_s)$ and $Q_0 = 1$ in equation (5), $f_{11}(\omega)$ and $f_{21}(\omega)$ are then defined as

$$\left.\begin{aligned} f_{11}(\omega) &= P(x_1, y_1, z_1, \omega)\big|_{Q_0=1} \\ f_{21}(\omega) &= P(x_2, y_2, z_2, \omega)\big|_{Q_0=1} \end{aligned}\right\}. \tag{8}$$

$f_{22}(\omega)$ is obtained in a similar manner.

It will be shown that $f_{11}(\omega)$ and $f_{22}(\omega)$ obtained from equation (8) are divergent when the point source model is used, and therefore they cannot be used to formulate four poles of three-dimensional cavities. Unlike the responses to point sources, responses $f_{11}(\omega)$ and $f_{22}(\omega)$ are bound and convergent when the surface source model is used.

Convergence Study

Calculations of pressure responses are made for a cubic cavity of length of 0.2 meters due to acoustic sources with unit volume flow rate. Other values for the material properties of acoustic medium are taken as $\rho_0 = 1.21\ (kg/m^3)$ and $c_0 = 343\ (m/s)$. Mass flow sources are on one of the boundary surfaces of the cavity along $x = L_x$ plane as shown in Figures 3 and 4. The point source and the center of surface source are placed at (L_x, $5L_y/9$, $5L_z/9$). The size of the surface source is taken as $b_s = c_s = 0.2/14\ (m)$.

Responses at Points away from a Source

First, pressure responses due to the point source and the surface source are calculated and compared at locations ($0.01L_x$, b_s, c_s) and ($0.99L_x$, b_s, c_s). The first point is relatively far away from the sources, and the second point is very close to the sources.

Figures 5(a) and 5(b) compares the pressure responses as the function of the number of natural modes used in the modal expansion solution by the two different source models. The responses are obtained for the field point far from the source at three frequencies (100, 500 and 1000 Hz). It is seen that responses using either source model converge to approximately the same values at each frequency. Figures 6(a) and 6(b) compares the responses for the second point (very close to the sources) at the same frequencies. The figure shows that two models, while both are bound, converge to completely different values. This is because the point source model cannot provide valid results in the near field.

Furthermore, for either source model, Figure 6 shows that a large number of modes must be used for the pressure response solution to converge in the near field, and the convergence of the solutions obtained from the point source model is especially slow.

Responses at Source Point

Figure 7(a) shows the pressure responses at the source point (x_s, y_s, z_s) obtained from the point source model as functions of the number of modes used in the calculation. The figure clearly shows that all the solutions diverge. In comparison, Figure 7(b) shows the pressure responses from the surface source model obtained at the center of the source surface as a function of the number of modes used in the modal expansion. It shows a clear convergence trend, although it is very slow. Convergence is observed at other points on the source surface as well.

Divergence of $f_{11}(\omega)$ and $f_{22}(\omega)$ Calculated from the Point Source Model

When the input source is modeled as a point source, the pressure response at the source point is equivalent to the following series for a given frequency:

$$\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{c_2}{l^2+m^2+n^2-c_1^2}, \tag{9}$$

where c_1 and c_2 are two constants. The first summation of the triple summation series (for n, e.g.) results in a finite value. The second summation (for m, e.g.) is actually the sum of infinite terms of finite values. Therefore, the series expressed in (9) diverges. From this finding, it is deducted that:

1. The surface source model is the minimum source geometry requirement for a three-dimensional acoustic system to have finite responses on the source.
2. The line source model is the minimum source geometry requirement for a two-dimensional acoustic system.

3. The point source model is only useful for a one-dimensional system to evaluate the pressure at the source point itself.

Because the series in modal expansion solution converges extremely slowly, the numerical calculation must be conducted very carefully. Otherwise the computer may ignore higher order terms due to its limitation in recognizing digits. A special scheme is used in this work when higher order terms are added. In the scheme, higher order terms are added until the sub-sum reaches a value sufficiently large for the computer to recognize in relation to the previous total sum, and then the sub-sum is added to the main series that consists of all the previous terms. In [1, 4], $f_{11}(\omega)$ and $f_{22}(\omega)$ are obtained based on the modal expansion method. However, the fact that these solutions are divergent was not recognized. It is probable that slow divergence of the series was not recognized. Also, the method was applied to annular cavities in [1]. In such a case, it is necessary to use Bessel functions, which makes accurate calculation of higher modes even more difficult.

It should be noted that divergence at the source point is not caused by the modal expansion method which was used to calculate pressure responses, but is caused by the inherent limitation of the point source model. This divergence problem associated with the point source in three- or two-dimensional cavities does not become an issue unless responses at the source point itself, or at a point very near to the source, are of interest. Unfortunately, the pressure response at source points ($f_{11}(\omega)$ or $f_{22}(\omega)$) must be found to formulate the four pole parameters using equation (2) as proposed in [1]. A modification of the procedure is proposed in the next section to overcome this difficulty.

New Method to Formulate Four Pole Parameters of Three-dimensional Cavities

A straightforward method to revise the definitions $f_{11}(\omega)$ and $f_{22}(\omega)$ is to use the fact that pressure responses over a surface source are always convergent. Therefore,

$$\begin{cases} f_{ii}(\omega) = \int_{\Gamma_s} \frac{P(r,\omega)}{u(r,\omega)} d\Gamma_s \Big/ A_s^2 \\ \int_{\Gamma_s} u(r,\omega) d\Gamma_s = 1 \end{cases} \quad i = 1,\ 2, \tag{10}$$

where $P(\mathbf{r},\omega)$ is the pressure response calculated based on the surface source model, Γ_s indicates the source surface, A_s is the area of the source surface, and $u(\mathbf{r},\omega)$ is the velocity distributed on the source surface. Assuming that the source has unit strength of volume flow rate and uniform distribution over the source surface, equation (10) can be simplified as

$$f_{ii}(\omega) = \int_{\Gamma_s} P(\mathbf{r},\omega) d\Gamma_s \Big/ A_s, \quad i = 1,\ 2. \tag{11}$$

Figure 8 shows the pressure distribution on the source surface at 100 Hz. One problem immediately observed in Figure 8 is that the pressure distribution on the source surface varies in a wide range, which makes the validity of the averaging process in equation (10) or (11) questionable. To avoid this problem, two pipes of the same section size as the input and output ports are added to the cavity, as shown in Figure 9. Pipe lengths are just long enough so that plane waves (uniform pressure across the section) can develop on the attached source planes. By doing this, errors involved in the averaging process can be minimized.

Obviously, a numerical method such as the BEM must be used for this method. In general, this is not a serious limitation because most three-dimensional cavities encountered in practice must be analyzed by a numerical method anyway. Another limitation is that the proposed method does not derive four poles of the cavity itself, but a system composed of the cavity and two one-dimensional pipes. This is also not a serious limitation because the four pole matrix of three-dimensional cavity must be combined with those of one-dimensional acoustic elements to form the system matrix.

Conclusions

It is shown that a numerical problem is encountered when four pole parameters of three-dimensional cavities are derived by the method proposed in [1]. In the procedure, the pressure responses which are necessary to formulate four poles are obtained based on the point source model. It is shown that the point source model is invalid for two- or three-dimensional cavities because of its singularity at the source point. The nature of singularity at the source point is studied. Theoretical and practical implications of the singularity at the source point are discussed. Necessary modifications of the original procedure are proposed to overcome this problem due to the singularity at the source point. The modifications are made by using the surface source model and extending the cavity to include two short pipes at the input and output ports.

References

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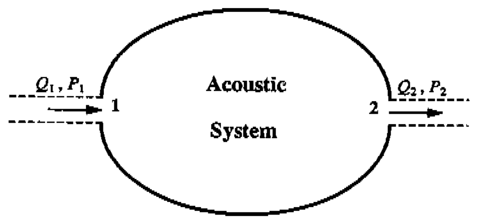

Figure 1. Three-dimensional acoustic system

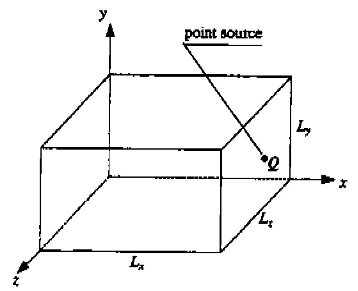

Figure 2. Rectangular cavity subjected to a surface source

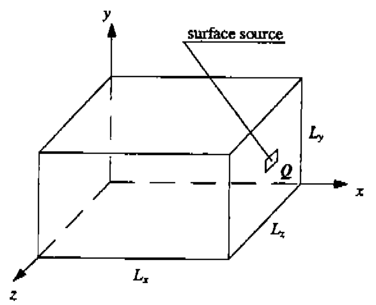

Figure 3. Rectangular cavity subjected to a surface source

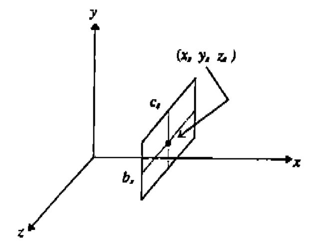

Figure 4. Source location and geometry
b_s and c_s are the dimensions of the source surface.
$(x_s\ y_s\ z_s)$ is the coordinate of the center of the surface source or the location of the point source.

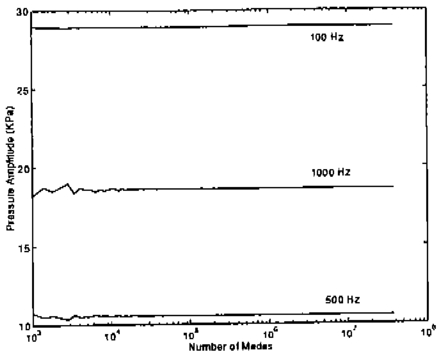

Figure 5(a) Response to surface source (far field)

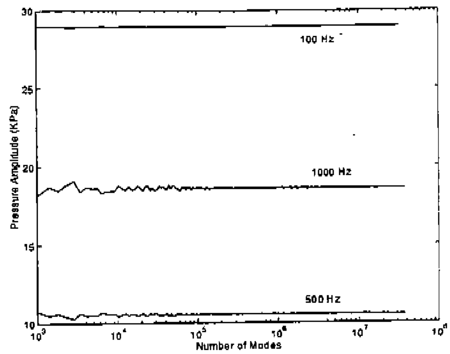

Figure 5(b) Response to point source (far field)

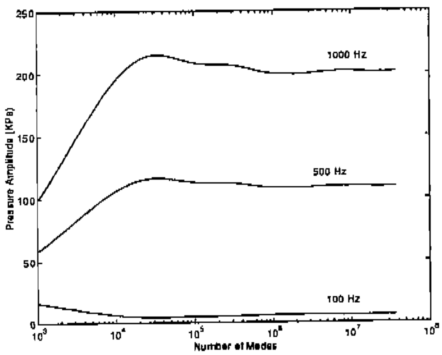

Figure 6(a) Response to surface source (near field)

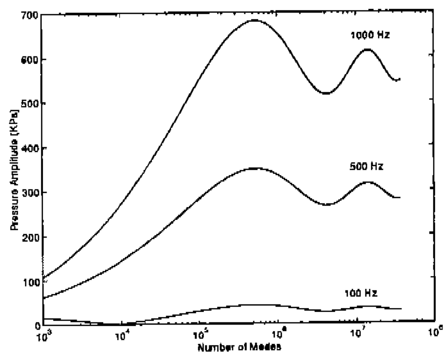

Figure 6(b) response to point source (near field)

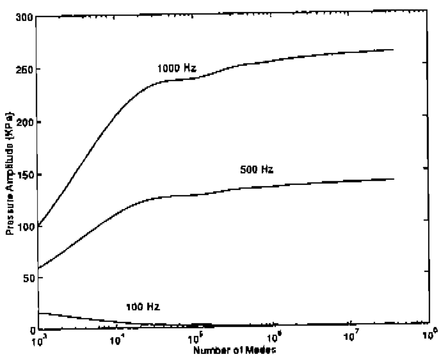

Figure 7(a) Response to surface source
(at the center of source surface)

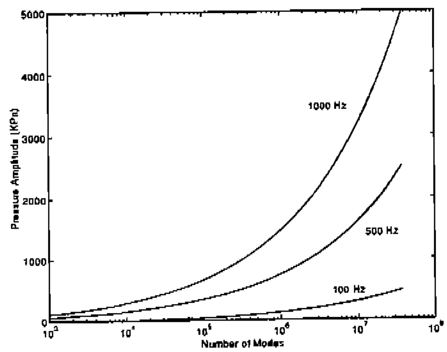

Figure 7(b) Response to point source
(at the source point)

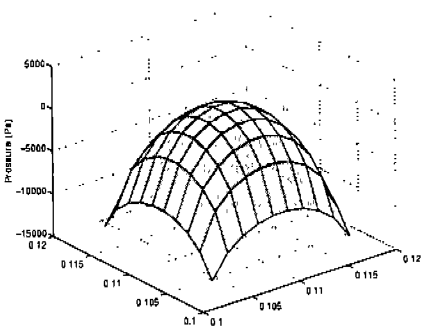

Figure 8. Pressure distribution on the
source surface (f = 100 Hz)

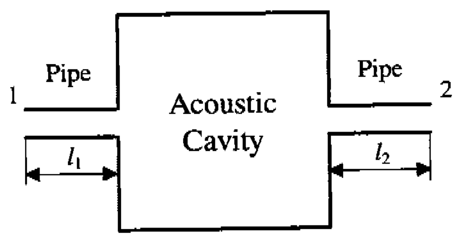

Figure 9. Extended acoustic system with
the cavity and two pipes